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Mathematics vs. Physics:
Ibn al-Haytham's Geometrical Conception of Space and the Refutation of Aristotle's Physical Definition of Place

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Epistemological reflections on foundational scientific principles become pivotal in the reformative development of specific branches of the sciences. The methodological adjustments that accompany such critical circumstances in the unfolding of scientific knowledge necessitate a reclassification of established concepts by way of accommodating novel theoretical hypotheses or emergent conceptual constructs. Progress within a given scientific discipline depends at times on radical reforms in methodology, which result in rethinking the epistemic models that are shared with other branches in science. To elucidate these dialectical dimensions in the evolution of innovative scientific rationalities, this study considers the phenomenon of 'the mathematisation of physics' in the context of history of the exact sciences in classical Islamic civilisation. This line of inquiry is specifically focused on the 11th century geometrical conception of space by the polymath al-Hasan ibn al-Haytham (Alhazen; d. ca. 1041 CE) and his refutation of Aristotle's physical definition of place.[1]

Mathematical Space

Ibn al-Haytham's geometrical conception of place (*al-makan*) as 'a mathematical *spatial extension*' was established in his *Discourse on Place* (*Qawl fi al-makan*),[2] which also rested on geometric demonstrations that grounded his rejection of the definition of *topos* (place) in Book *Delta* (IV) of Aristotle's *Physics*.

Ibn al-Haytham endeavoured to present his geometrical conception of *al-makan* (place) as a solution to a longstanding problem that remained philosophically unresolved, which, to our knowledge, constituted in its own right the first viable attempt to mathematise 'place' in history of science.

Ibn al-Haytham aimed primarily at promoting his geometrical conception of place as 'spatial extension' in an attempt to address selected mathematical problems that emerged in reference to unprecedented developments in geometrical transformations (*al-naql*; like similitude, translation, homothety, affinity), the introduction of motion (*al-haraka*; *kinesis*) in geometry, the use of geometric projections in spherics, and the anaclastic properties of conic, cylindrical, and spherical sections; all undertaken within the 9th-10th century prolongations of the Apollonian-Archimedean legacy in mathematics — To mention in this context the research of polymaths of the calibre of the Banu Musa, Thabit ibn Qurra, Ibrahim ibn Sinan, al-Khazin, al-Quhi, al-Sijzi, and Ibn Sahl.

Besides the epistemic tendency to offer mathematical solutions to problems in theoretical philosophy, Ibn al-Haytham's endeavour in geometrising place was undertaken in view of grounding his own research in mathematical analysis and synthesis (*al-tahlil wa-al-tarkib*; with implications on the development of infinitesimal mathematics),[3] and in support of his studies on knowable mathematical entities (*al-ma'lumat*). Ibn al-Haytham also aimed at reorganising most of the notions of geometry and rethinking them anew in terms of motion, and by way of positing an abstract spatial domain that receives geometrical transformations.[4] Consequently, he had to critically reassess the dominant philosophical conceptions of place in his age, which were encumbered by inconclusive theoretical disputes that were principally developed in reaction to Aristotle's *Physics*.

Physical Place

Aristotle defined *topos* (place) as: 'the innermost primary surface-boundary of the containing body that is at rest, and is in contact with the outermost surface of the mobile contained body' (*Physics*, IV, 212a 20-21).[5] The *makan* of the Aristotelian *falasifa* or *hukama'* consisted of a *sath muhit* or *sath hawi* (a surrounding surface or a containing enveloping boundary).

In contesting this *physical* conception of *topos*, Ibn al-Haytham posited *al-makan* as a postulated void (*khala' mutakhayyal*), whose existence is secured in the imagination, like it is the case with invariable geometrical entities. He also held that this 'postulated void' consisted of imagined immaterial distances that are between the opposite points of the surfaces surrounding it. He furthermore noted that the imagined distances of a given body, and those of its containing place, get superposed and united in such a way that they become the same distances as mathematical lines having lengths without widths.

Ibn al-Haytham's geometrical conception of place as a relational extension was 'ontologically' neutral.[6] His mathematical notion of *al-makan* was not simply obtained through a 'theory of abstraction' as such, nor was it derived by way of a 'doctrine of forms', nor was it grasped as being the (phenomenal) 'object' of 'immediate experience' or 'common sense'. Rather, his geometrised place resulted from a mathematical isometric 'bijection' function between two sets of relations or distances.[7] Nothing is thus retained of the properties of a body other than *extension*, which consists of mathematical distances. Accordingly, the *makan* of a given object is a 'region of extension' that is defined by the distances between its points, on which the distances of that object can be applied 'bijectively'.[8]

It is worth noting here that Aristotle's definition of place received bold classical critiques in the commentaries on his work, including the objections raised by Philoponus in defence of the conception of *topos* as interval (*diastasis*; *diastema*).[9] However, what primarily distinguishes Ibn al-Haytham from his predecessors is that his critique of Aristotle was mathematical, and, that it was partly auxiliary to his own response to the epistemic need to geometrise place, while those who came before him restricted their objections to the Aristotelian notion of *topos* within the domain of philosophical deliberations in classical physics.

Geometrical Demonstrations

To offer some highlights of Ibn al-Haytham's geometrical demonstrations in rejecting Aristotle's definition of *topos*, let us consider the case of a parallelepiped (*mutawazi al-sutuh*; a geometric solid bound by six parallelograms) that occupies a given place delimited by the surfaces enclosing it. If that parallelepiped were to be divided into two parts by a plane that is parallel to one of its surfaces, and is then recomposed, the cumulative size of the parts resulting from its partition would be equal to the magnitude of that parallelepiped prior to being divided, while the total sum of the surface areas of the parts would be greater than that of the parallelepiped prior to its division. Following the Aristotelian definition, and in reference to this partitioned parallelepiped, one would conclude that: an object divided into two parts occupies a place that is larger than the one it occupied prior to its division. Hence, 'the place of a given body increases while that body does not, and an object of a given magnitude is contained in unequal places'; which is an untenable proposition.[10]

Likewise, if we consider the case of a parallelepiped that we carve with carefully selected geometrical shapes, we would diminish its bodily magnitude while the total sum of its surface areas would increase. Following the Aristotelian definition, and in reference to this carved parallelepiped, one would conclude that: 'an object that diminishes in size occupies a larger place prior to its diminution in magnitude'; which constitutes an indefensible thesis. Moreover, using mathematical demonstrations, in reference to geometrical solids of equal surface-areas, which are based on studies conducted on figures that are of equal perimeters, Ibn al-Haytham demonstrated that 'the sphere is the largest in size with respect to all other solids that have equal areas of their enveloping surfaces' (*al-kura a'zam al-ashkal al-lati ihatatuha mutasawiya*). Ultimately, the volumetric magnitude of geometric solids remains the same despite changes in their shape (like when modelling a given piece of wax into the shape of a sphere, and then giving it the form of a cylinder, the quantifications of its material volume and the magnitude of its spatial extension remain the same, while its total surface area diminishes when it is transformed from a spherical shape into a cylindrical one).

The geometrical place of a given object is posited as a 'metric' of a region of 'mathematical *space*', which is occupied by a given body that is conceived extensionally, and corresponds with its own geometrical place by way of 'isometric bijection'. The epistemological and historical validity of Ibn al-Haytham's geometrisation of place was ultimately confirmed in the maturation of mathematics and science in the 17th century conceptions of extension *qua* space; particularly in reference to the works of Descartes and Leibniz.[11] Furthermore, the prolongations of Euclidean geometry benefited from the geometrisation of place, which among other developments resulted in the emergence of what came to be known in periods following Ibn al-Haytham's age as: 'Euclidean space'; namely, an appellation that is coined in relatively modern times, and describes a notion that is historically posterior to the geometry of figures as embodied in Euclid's *Elements* (*Kitab Uqlidis fi al-usul*).[12] After all, the term deployed by Euclid that is closest to a notion of '*space*' (*espace*; *Raum*), as expressed in the Greek appellation: '*Khora*', is: '*Khorion*' (*Data*, Prop. 55; *Elements* VI, Prop. 25), which designates 'an area enclosed within the perimeter of a specific geometric abstract figure'.[13]

Philosophical Critique

Ibn al-Haytham's conception of place was eventually criticised by the Aristotelian philosopher 'Abd al-Latif al-Baghdadi (fl. 13th cent.) in a treatise titled: *Fi al-radd 'ala Ibn al-Haytham fi al-makan* (which consisted of an attempted refutation of Ibn al-Haytham's geometrical definition of place).[14]

Baghdadi argued that Ibn al-Haytham did not logically account for a correspondence/concomitance between a given object and its 'place' (*qua* 'enveloping surfaces') as both being subject to change.[15] If a given object changes by way of division/partition and/or diminution in size, its place changes as well, due to the transformation of its shape and its associated surface areas. To explore this proposition, let us reconsider the case of the parallelepiped which was divided and/or carved; in both instances it has been transformed in its shape and associated surfaces, hence its place changed as well. If a divided object becomes two distinct entities, then its shape is likewise transformed into two separate shapes, and its original place is transmuted into two different places with distinct surface areas. The fact that the parallelepiped is divided or carved entails that it is no longer the same entity that it was prior to its division or carving; and so is the case with its place, shape and the total sum of its surface areas, which get transformed into something else. According to Baghdadi, Ibn al-Haytham's geometrical proofs neglected the fact that a change in a given object leads to a transformation in its shape, the total sum of its surface areas, and the place it occupies. Failing to recognise that the parallelepiped becomes something other than itself, when partitioned or carved, results in neglecting the fact that its shape, place, and the total sum of its surface areas are also transformed. It is hence valid to say that an object occupies a different place when it is divided and/or carved, given that it is no longer the same object *per se*, but is rather transformed into another sort of entity.

In all of this, Baghdadi presupposed philosophical accounts of the individuation of bodies as a modality by virtue of which he attempted to offer counterexamples to Ibn al-Haytham's geometrical demonstrations, while also erroneously assuming that the latter's propositions were reducible to one and the same type of arguments. Moreover, Baghdadi wondered how the *actual* distances (*bi-al-fi'l*) of a given body are superposed and united with the imagined *potential* distances (*bi-al-quwwa*) of its place. He was unsure whether Ibn al-Haytham considered the distances of a body and those of its place as being potentialities and not actualities; hence positing them as non-existents. He furthermore rejected the claim that the presumably 'superposed distances' (*al-ab'ad al-mutatabiqa*) can be actual existents, since this implies a co-penetration of material entities;[16] hence failing to recognise the epistemic entailments of Ibn al-Haytham's mathematisation of place as geometric *extension*.

Baghdadi asserted also that the mathematician judges distances insofar that they are imagined in the mind as being abstracted from matter (*mutakhayyala fi al-dhihn*), while the physicist grasps them as existing externally (*mawjuda fi al-kharij*). Yet, the difference between the research of the *physicist* and that of the *mathematician* did not only reflect a binary contrast between an Aristotelian metaphysics/physics and a Platonist theory of forms, it rather pointed also to a 'third' classical tradition that was 'Archimedean', which was not satisfied with the mere philosophical cognition of 'natural phenomena', but essentially aimed at investigating them mathematically. It is this third epistemic pathway that inspired Ibn al-Haytham's 'geometrisation of place' and embodied his scientific reform in 'mathematising physics'.

NOTES

[1] This essay rests on some of my earlier publications, including the following articles: Nader El-Bizri, 'Epistolary Prolegomena: On Arithmetic and Geometry', in *Epistles of the Brethren of Purity. The Ikhwan al-Safa' and their Rasa'il: An Introduction*, ed. Nader El-Bizri (Oxford: Oxford University Press, in association with The Institute of Ismaili Studies, 2008), pp. 180-213; Nader El-Bizri, '*Le problème de l'espace*: Approches optique, géométrique et phénoménologique', in *Oggetto e spazio. Fenomenologia dell'oggetto, forma e cosa dai secoli XIII-XIV ai post-cartesiani*, ed. Graziella Federici Vescovini and Orsola Rignani, *Micrologus Library 24* (Firenze: SISMEL, Edizioni del Galluzzo, 2008), pp. 59-70; Nader El-Bizri, 'In Defence of the Sovereignty of Philosophy: al-Baghdadi's Critique of Ibn al-Haytham's Geometrisation of Place', *Arabic Sciences and Philosophy*, Vol. 17, Issue 1 (2007), pp. 57-80; Nader El-Bizri, 'A Philosophical Perspective on Alhazen's *Optics*', *Arabic Sciences and Philosophy*, Vol. 15, Issue 2 (2005), pp. 189-218; Nader El-Bizri, 'La perception de la profondeur: Alhazen, Berkeley et Merleau-Ponty', *Oriens-Occidens: sciences, mathématiques et philosophie de l'antiquité à l'âge classique (Cahiers du Centre d'Histoire des Sciences et des Philosophies Arabes et Médiévales, CNRS)*, Vol. 5 (2004), pp. 171-184.

[2] For the Arabic critical edition and annotated French translation of this treatise (*Fi al-makan*; *Traité sur le lieu*) see: Roshdi Rashed, *Les mathématiques infinitésimales du IXe au XIe siècle, Volume IV: Ibn al-Haytham, méthodes géométriques, transformations ponctuelles et philosophie des mathématiques* Vol. 4 (London: al-Furqan Islamic Heritage Foundation, 2002), pp. 666-685.

[3] For the Arabic critical edition and annotated French translation of this treatise (*Fi al-ta°lil wa-al-tarkib*; *L'Analyse et la synthèse*) see: Rashed, *Les mathématiques infinitésimales*, Vol. 4 (2002), pp. 230-391.

[4] For the Arabic critical edition and annotated French translation of this treatise (*Fi al-maʿlumat*; *Les connus*) see: Rashed, *Les mathématiques infinitésimales*, Vol. 4 (2002), pp. 444-583.

[5] Aristotle, *Physics*, ed. W. David Ross (Oxford: Oxford University Press, 1936).

[6] I have investigated this ontological question elsewhere in the following articles: Nader El-Bizri, '*ON KAI KH³RA*: Situating Heidegger between the *Sophist* and the *Timaeus*', *Studia Phaenomenologica*, Vol. IV, Issue 1-2 (2004), pp. 73-98; Nader El-Bizri, '*Ontopoiesis* and the Interpretation of Plato's *Khora*', *Analecta Husserliana: The Yearbook of Phenomenological Research*, Vol. LXXXIII (2004), pp. 25-45; Nader El-Bizri, 'A Phenomenological Account of the *Ontological Problem of Space*', *Existentia Meletai-Sophias*, Vol. XII, Issue 3-4 (2002), pp. 345-364; Nader El-Bizri, '*Qui-etes vous Khora?* Receiving Plato's *Timaeus*', *Existentia Meletai-Sophias*, Vol. XI, Issue 3-4 (2001), pp. 473-490.

[7] 'Bijectivity' describes a mathematical 'one-to-one correspondence' function (f) that is from a given set X to a given set Y such as for every y in Y, there is exactly one x in X such as $f(x) = y$.

[8] Rashed, *Les mathématiques infinitésimales*, Vol. 4 (2002), pp. 658, 901.

[9] *Simplicii in Aristotelis Physicorum Libros Quattuor Priores Commentaria*, ed. H. Diels in *Commentaria in Aristotelem Graeca*, Vol. IX (Berlin, 1882); Simplicius, *Corollaries on Place and Time*, trans. J. O. Urmson (London: Duckworth, 1992); 601,25-611,10; 604,5-11. See also: Simplicius, *On Aristotle, Physics 4.1-5, 10-14*, trans. J. O. Urmson (London: Duckworth, 1992); Philoponus, *Corollaries on Place and Void*, and: Simplicius, *Against Philoponus on the Eternity of the World*, trans. D. Furley and C. Wildberg (London: Duckworth, 1991).

[10] Rashed, *Les mathématiques infinitésimales*, Vol. 4 (2002), pp. 670-673.

[11] René Descartes, *Discours de la méthode*, in *Œuvres de Descartes*, eds. Charles Adam and Paul Tannery (Paris: Vrin, 1965), Vol. 6, p. 36; Gottfried Wilhelm Leibniz, *La Caractéristique géométrique*, ed. Javier Echeverria, trans. Marc Parmentier (Paris: Vrin, 1995), p. 235.

[12] Euclid, *The Thirteen Books of Euclid's Elements*, vols. 1-3, translated with introduction and commentary by Thomas L. Heath (New York: Dover Publications, 1956), 2nd edition. The Greek edition of Euclid's *Elements* is preserved in the Teubner Classical Library, 8 vols. with a supplement, titled: *Euclides opera omnia*, eds. J. L. Heiberg and H. Menge (Leipzig, 1883-1916).

[13] As for instance noted in Euclid's *Data* (*Dedomena*; *al-Muʿtayat*) Prop. 55 (related to: *Elements*, VI, Prop. 25): 'if an *area* [*Khorion*] be given in form and in magnitude, its sides will also be given in magnitude'.

[14] For the Arabic edition and annotated French translation of this treatise (*Fi al-radd ʿala Ibn al-Haytham fi al-makan*; *La réfutation du lieu d'Ibn al-Haytham*) see: Rashed, *Les mathématiques infinitésimales*, Vol. 4 (2002), pp. 908-953.

[15] Rashed, *Les mathématiques infinitésimales*, Vol. 4 (2002), pp. 914-915.

[16] Rashed, *Les mathématiques infinitésimales*, Vol. 4 (2002), pp. 916-917.